Unit - VII

LINEAR ALGEBRA - 1

7.1 Introduction

Linear Algebra broadly deals with theoretical and practical applications of linear transformations, linear system of equations etc. ***Matrix Theory*** is an important topic in linear algebra and the reader is acquainted with this to a certain extent. In this unit we discuss matrix oriented topics such as rank of a matrix, elementary transformation consistency of a system of linear algebraic equations using the concept of rank of a matrix, solutions of a linear algebraic system of equations by some special methods.

7.2 Recapitulation of the basic matrix theory

A set of mn elements written in an array of m rows and n columns embedded in brackets [ ] or ( ) is called a ***matrix*** of order $m \times n$ (m by n). The following array is a typical $m \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots a_{mn} \end{bmatrix} = [a_{ij}].$$

Order of $A = O(A) = m \times n$

If $m = n$ the matrix is called a ***square matrix*** of order n

The matrix of order $1 \times n$ is called a ***row matrix*** and a matrix of order $n \times 1$ is called a ***column matrix***

Examples : $[1 \ 2 \ 3]$ ... Row matrix (order 1×3)

$$\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \quad \ldots \text{ column matrix (order } 3 \times 1\text{)}$$

Consider a square matrix A of order n as follows

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots a_{nn} \end{bmatrix}$$

In A the elements $a_{11}, a_{22}, \ldots a_{nn}$ constitute *principal diagonal* of the square matrix A.

The sum of all these elements is called the *trace* of A. A square matrix having all the elements below the principal diagonal zero is called an ***upper triangular matrix*** and a square matrix having all the elements above the principal diagonal zero is called a ***lower triangular matrix***

Examples : $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 8 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 4 & 5 & 0 \\ 6 & 7 & 8 \end{bmatrix}$ are respectively upper and lower triangular matrices.

A square matrix is said to be a ***diagonal matrix*** if every element other than the principal diagonal elements are zero.

Examples : $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

If every element of a diagonal matrix is the same then it is called a ***scalar matrix.***

Examples : $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

A scalar matrix in which each diagonal element is equal to unity, that is 1 is called a ***Unit matrix*** or ***Identity matrix*** usually denoted by I.

Examples : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A matrix having all its elements equal to zero is called a ***null matrix.***

Examples : $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

The matrix obtained by interchanging their rows and columns is called as the ***transpose*** of the given matrix usually denoted by A' where A is the given matrix.

Obviously $(A')' = A$.

Examples : If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 4 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 4 \end{bmatrix}$

If $A = [2 \quad -1 \quad 4]$ then $A' = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

A square matrix A is said to be ***symmetric*** if $A = A'$ and ***skew symmetric*** if $A = -A'$.

Examples : $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 4 & 2 \\ 4 & 5 & 0 \\ 2 & 0 & 7 \end{bmatrix}$ are symmetric matrices.

(observe that row elements and column elements are the same)

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix}$ are skew symmetric matrices.

(observe that the principal diagonal elements are zero and column elements are negative of the row elements.)

Algebra of matrices

If A is any matrix and k be any scalar the matrix obtained by multiplying every element of A by k is called the scalar multiple of the matrix A denoted by kA.

Examples :

If $A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 4 & 6 \end{bmatrix}$ then $3A = \begin{bmatrix} 6 & 9 & 15 \\ -3 & 12 & 18 \end{bmatrix}$, $\frac{1}{2} A = \begin{bmatrix} 1 & 3/2 & 5/2 \\ -1/2 & 2 & 3 \end{bmatrix}$

Sum and Difference of two matrices

If A and B are two matrices of the same order, their sum $A + B$, difference $A - B$ is the sum/difference of the corresponding elements.

Examples

If $A = \begin{bmatrix} -1 & 4 \\ 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 3 \\ 5 & -4 \end{bmatrix}$ then

$A + B = \begin{bmatrix} 5 & 7 \\ 10 & 0 \end{bmatrix}$, $A - B = \begin{bmatrix} -7 & 1 \\ 0 & 8 \end{bmatrix}$

$2A + 3B = \begin{bmatrix} 16 & 17 \\ 25 & -4 \end{bmatrix}$, $A + \frac{1}{2} B = \begin{bmatrix} 2 & 11/2 \\ 15/2 & 2 \end{bmatrix}$

Product of two matrices

If A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$ then the product AB exists and will be a matrix of order $m \times p$.

Row elements of A are multiplied with the corresponding column elements of B and are added.

Examples :

1. Consider $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 6 \end{bmatrix}$

Order of A, $O(A) = 2 \times 3$, $O(B) = 3 \times 2$ $\therefore$ $O(AB) = 2 \times 2$

The product AB is computed as follows.

$$AB = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3\times1 + 2\times0 + 1\times3, & 3\times2 + 2\times4 + 1\times6 \\ 2\times1 + 4\times0 + 2\times3, & 2\times2 + 4\times4 + 2\times 6 \end{bmatrix} = \begin{bmatrix} 6 & 20 \\ 8 & 32 \end{bmatrix}$$

Also let us examine the possibility of the computation of the matrix product BA

$O(B) = 3 \times 2$, $O(A) = 2 \times 3$ $\therefore$ $O(BA) = 3 \times 3$

The product BA is computed as follows.

$$BA = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1\times3+2\times2, & 1\times2+2\times4, & 1\times1+2\times2 \\ 0\times3+4\times2, & 0\times2+4\times4, & 0\times1+4\times2 \\ 3\times3+6\times2, & 3\times2+6\times4, & 3\times1+6\times2 \end{bmatrix}$$

ie., $$BA = \begin{bmatrix} 7 & 10 & 5 \\ 8 & 16 & 8 \\ 21 & 30 & 15 \end{bmatrix}$$

Remark : *We will come across with product of two square matrices of the same order. Obviously the resulting product matrix will also be a square matrix of the same order.*

Note : Properties of matrix multiplication.

1. The matrix product AB may exist but BA may not exist. Even if both exist, $AB \neq BA$. Matrix multiplication in general is not commutative.

2. If $O(A) = m \times n$, $O(B) = n \times p$, $O(C) = p \times q$ then $A(BC) = (AB)C$ which implies that the matrix multiplication is associative.

3. If $O(A) = m \times n$ and $O(B) = n \times p = O(C)$ then $A(B+C) = AB + AC$, which implies that matrix multiplication is distributive.

4. $(AB)' = B'A'$

5. If A is any square matrix, the matrix product $A \cdot A$ is denoted by A^2. Further $A \cdot A^2 = A^3$ and so-on.
   In general if p and q are positive integers,
   $A^p \cdot A^q = A^{p+q}$ and $(A^p)^q = A^{pq}$.

A square matrix A is said to be ***orthogonal*** if $AA' = I = A'A$

A square matrix A is said to be ***singular*** if its determinant is zero and A is said to be ***nonsingular*** if its determinant is not equal to zero.

That is, $|A| = 0 \Rightarrow A$ is singular and

$|A| \neq 0 \Rightarrow A$ is nonsingular.

If A and B are two square matrices of the same order such that

$$AB = BA = I$$

then B is called the inverse of A denoted by A^{-1}.

The necessary and the suffecient condition for a square matrix A to possess inverse is that $|A| \neq 0$. That is to say that the inverse of a square matrix exists if and only if the matrix A is nonsingular.

Note : 1. *The inverse of a square matrix is unique.*

2. $(AB)^{-1} = B^{-1}A^{-1}$

3. *We have already said that* $AA' = I = A'A \Rightarrow A$ *is orthogonal.*

Comparing this with the definition of the inverse of a square matrix we can also conclude that

$$A' = A^{-1} \Rightarrow A \text{ is orthogonal.}$$

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the adjoint of A denoted by Adj A is given by

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}.$$

Here A_{ij} represents the cofactor of a_{ij} in $|A|$. That is, $A_{ij} = (-1)^{i+j}$ multiplied by the value of the determinant obtained by deleting the i^{th} row and j^{th} column. However in the case of a second order square matrix, the cofactors will be a single element.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then Adj $A = \begin{bmatrix} +a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Now, for the third order square matrix A we have,

$$\text{Adj } A = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

Remark : *It is preferable to write the cofactors of elements column wise premultiplied by the signs* $+, -, +, -, +, -, + \cdots$ *and enter row wise in the matrix, resulting in the adjoint of the given matrix.*

Further the adjoint of a square matrix helps in finding the inverse of a nonsingular matrix A by the following established result.

$$A^{-1} = \frac{1}{|A|} \text{ Adj } A$$

Illustrative Examples to compute the inverse of a given square matrix.

1. $A = \begin{bmatrix} 1 & -2 \\ -3 & 8 \end{bmatrix}$

>> $|A| = \begin{vmatrix} 1 & -2 \\ -3 & 8 \end{vmatrix} = 8 - (+6) = 2$

$\text{Adj } A = \begin{bmatrix} +8, & -(-2) \\ -(-3), & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 8 & 2 \\ 3 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

>> $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = 1\ (-28+30) - 0 - 1\ (-18-0) = 20$

$$\text{Adj } A = \begin{bmatrix} +(-28+30), & -(0-6), & +(0+4) \\ -(-21-0), & +(-7-0), & -(5+3) \\ +(-18-0), & -(-6-0), & +(4-0) \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

7.3 Elementary transformations (operations) associated with a matrix

The following are the elementary ***row transformations*** of a matrix. The transformations can also be applied for columns.

1. Interchange of any two rows.
2. Multiplication of any row by a non zero constant.
3. Addition to any row a constant multiple of any other row.

Elementary transformations along with the notations we use is illustrated in the following table by considering the matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Example

	Elementary row transformation	Notation	Resultant of matrix A
1.	Interchange of first and second row	$R_1 \leftrightarrow R_2$	$\begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$
2.	Multiplication of third row by a constant k	kR_3	$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ kc_1 & kc_2 & kc_3 \end{bmatrix}$
3.	Addition to second row k times the first row	$R_2 \rightarrow kR_1 + R_2$	$\begin{bmatrix} a_1 & a_2 & a_3 \\ (ka_1+b_1) & (ka_2+b_2) & (ka_3+b_3) \\ c_1 & c_2 & c_3 \end{bmatrix}$

Equivalent matrices : Two matrices A and B of the same order are said to be ***equivalent*** if one matrix can be obtained from the other by a finite number of successive elementary row (column) transformations. We use the notation $A \sim B$.

A non zero matrix A is said to be in ***row echelon form*** if the following conditions prevail.

(a) All the zero rows are below non zero rows.

(b) The first non zero entry in any non zero row is 1.

(*Obviously in view of the condition (a) the entries below 1 in the same column are zero.*)

Example :

1. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The given matrix A is reduced to an echelon form first by applying a series of elementary row transformations.

Later column transformations are performed to reduce the matrix to one of the following four forms, called the **Normal form** of A.

(i) I_r (ii) $[\, I_r\,,\ 0\,]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

where I_r is the identity matrix of order r.

Observe the following corresponding examples of the normal form of a given matrix.

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

These are equivalent to the following forms respectively.

(i) I_3 (ii) $[\, I_3\,,\ 0\,]$ (iii) $\begin{bmatrix} I_2 \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

- In order to reduce the given matrix to a row echelon form we must prefer to have the leading entry (*first entry in the first row*) non zero, much prefarably 1.
- In the case when this entry is zero we can interehange with any suitable row to meet the requirement.
- We then focus on the leading non zero entry (*starting from the first row*) to make all the elements in that column zero. However the transformation has to be performed for the entire row.

- Row echelon form will be achieved first and we have to continue further to arrive at the normal form
- We need to perform column transformations in the same way to achieve the normal form.

Note : ***It is advisible to avoid fractions as far as possible during the process of elementary transformations.***

WORKED PROBLEMS

1. *Reduce the following matrix to the echelon form*

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & 8 \end{bmatrix}$$

\>> Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & 8 \end{bmatrix}$

$R_2 \to 2R_1 + R_2, \; R_3 \to -R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & 8 \end{bmatrix}$$

$R_3 \to 1/4 \cdot R_2 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9/4 & 8 \end{bmatrix}$$

$1/8 \cdot R_2, \; 4/9 \cdot R_3$

$\therefore \quad A \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 1 & 32/9 \end{bmatrix}$ **is the row echelon form of A**

---

2. *Applying elementary transformations, reduce the following matrix to the normal form.*

$$\begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

\>> Let $A = \begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 3 & 2 & 5 & 7 & 12 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

$$R_2 \to -3R_1 + R_2, \quad R_3 \to -3R_1 + R_3$$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(We have to perform column transformations to reduce to the normal form)

$$C_2 \to -C_1 + C_2, \quad C_3 \to -2C_1 + C_3, \quad C_4 \to -3C_1 + C_4, \quad C_5 \to -5C_1 + C_5$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \to -C_2 + C_3, \quad C_4 \to -2C_2 + C_4, \quad C_5 \to -3C_2 + C_5$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(-1)\, C_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ **is the normal form of A.**

---

3. *Find the normal form of the following matrix*

>> Let $$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & 0 & 2 & 6 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$$R_2 \to -2R_1 + R_2$$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & 4 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$R_3 \to 1/2 \cdot R_2 + R_3$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, $C_2 \to -1/2 \cdot C_1 + C_2,\ C_3 \to -3/2 \cdot C_1 + C_3,\ C_4 \to -1/2 \cdot C_1 + C_4$

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to -2\,C_2 + C_3,\ C_4 \to 2\,C_2 + C_4$

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1/2 \cdot C_1,\ -1/2 \cdot C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ **is the normal form of A.**

---

4. *By performing elementary row and column transformations, reduce the following matrix to the normal form.*

$$\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

>> Let
$$A = \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 1 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2, \; R_4 \to -4R_1 + R_4$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -4 & -2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$$

$R_4 \to -R_2 + R_4$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -4 & -2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 1 & 9 & -4 \end{bmatrix}$$

$R_4 \to -R_3 + R_4$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -4 & -2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, $C_2 \to 2C_1 + C_2, \; C_3 \to -C_1 + C_3, \; C_4 \to 4C_1 + C_4, \; C_5 \to 2C_1 + C_5$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_2 + C_3, \; C_4 \to -3C_2 + C_3, \; C_5 \to -C_2 + C_5$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to -9\,C_3 + C_4,\ \ C_5 \to 4\,C_3 + C_5$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$ **is the normal form of A.**

3. Reduce the following matrix to normal form

$$\begin{bmatrix} 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 9 & 3 & 1 & 0 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

>> Let
$$A = \begin{bmatrix} 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 9 & 3 & 1 & 0 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & -6 \\ 9 & 3 & 1 & 0 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

$R_2 \to -3\,R_1 + R_2,\ \ R_3 \to -9\,R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & -6 & -8 & -9 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

$R_3 \to -2\,R_2 + R_3,\ \ R_4 \to -2\,R_2 + R_4$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 5 & 27 \end{bmatrix}$$

$R_4 \to 5/4 \cdot R_3 + R_4$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & 153/4 \end{bmatrix}$$

Now, $C_2 \to -C_1 + C_2,\ C_3 \to -C_1 + C_3,\ C_4 \to -C_1 + C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & 153/4 \end{bmatrix}$$

$$C_3 \to -2/3 \cdot C_2 + C_3,\ C_4 \to -3\, C_2 + C_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & 153/4 \end{bmatrix}$$

$$C_4 \to 9/4 \cdot C_3 + C_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 153/4 \end{bmatrix}$$

Finally, $-1/3 \cdot C_2,\ -1/4 \cdot C_3,\ 4/153 \cdot C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4 \text{ is the normal form of } A.$$

7.5 Rank of a matrix

The ***rank*** of a matrix A **in echelon form** is equal to the ***number of non zero rows.*** It is denoted by $\rho(A)$.

Any matrix A of order $m \times n$ can be reduced to one of the normal forms :

(i) I_r (ii) $[I_r,\ 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

It is evident that the matrix in these normal forms will have r non zero rows.

Hence $\rho(A) = r$

If A is a $m \times n$ matrix of rank r, then there exists nonsingular matrices P and Q such that the matrix $P\,A\,Q$ is in the normal form.

Further we can say that if r is the rank of a matrix A of order $m \times n$ $(r \le m)$, r number of rows of the matrix are linearly independent.

Observe the following examples.

1. $$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Performing, $R_2 \to -3\,R_1 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \rho\,(A) = 1$$

This means that *one row* of the matrix *A is linearly independent*. It can be easily seen that $R_2 = 3\,R_1$ or $R_1 = 1/3\ R_2$

2. $$A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 1 & 5 & 7 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ yields,

$$A \sim \begin{bmatrix} -1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$$

$R_2 \to 2\,R_1 + R_2$, $R_3 \to R_1 + R_3$ gives us

$$A \sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 7 & 10 \\ 0 & 7 & 10 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$ gives

$$A \sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 7 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

Finally $-R_1$ and $1/7 \cdot R_2$ yields

$$A \sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 10/7 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho\,(A) = 2$$

This means that *two rows* of the matrix *A are linearly independent*. It can be easily seen that $R_1 + R_2 = R_3$ or $R_2 = R_3 - R_1$ or $R_1 = R_3 - R_2$.

Any one of the row of the matrix A is expressible in terms of the other two rows.

The application of the rank of a matrix is discussed in the next article.

Remark : ***We can write down the rank of the matrices in problems 1 to 5 discussed earlier.***

Problem No.	1	2	3	4	5
$\rho(A)$	3	2	2	3	4

WORKED PROBLEMS

Find the rank of the following matrices by applying elementary row transformations [6 to 9]

6. $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

>> $R_2 \to -2R_1 + R_2 \quad R_3 \to -R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_2 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(-1)R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A in the row echelon form is having ***two*** non zero rows.

Hence the rank of A is 2. Thus $\rho(A) = 2$

7. $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$

>> $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

$R_3 \to -2R_1 + R_3$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$R_3 \rightarrow -R_2 + R_3$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1/2 \cdot R_1,\ 1/2 \cdot R_2$

$$A \sim \begin{bmatrix} 1 & 3/2 & 5/2 & 2 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A in the row echelon form is having *two* non zero rows.

Thus $\rho(A) = 2$

[illegible]

\>> $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \rightarrow -2R_1 + R_2,\ R_3 \rightarrow -R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -9 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & -5 & -9 & 1 \end{bmatrix}$$

$R_3 \rightarrow 2R_2 + R_3,\ R_4 \rightarrow 5R_2 + R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$ and $-1/4 \cdot R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A in the row echelon form is having ***three*** non zero rows.

Thus $\rho(A) = 3$

>> [*Firstly we prefer to interchange the first and second rows as it would be convenient to make the leading entry in the other rows zero*]

$R_2 \leftrightarrow R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2,\ R_3 \to -R_1 + R_3,\ R_4 \to -R_1 + R_4$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_2 + R_3,\ R_4 \to R_2 + R_4$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -6 & -7 \end{bmatrix}$$

$R_4 \to -2R_3 + R_4$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$1/2 \cdot R_1, \quad 1/2 \cdot R_2, \quad -1/3 \cdot R_3$

$$A \sim \begin{bmatrix} 1 & 1/2 & 3/2 & 2 \\ 0 & 1 & 2 & 7/2 \\ 0 & 0 & 1 & 4/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All the *four* rows are non zero in the row echelon form A, **Thus** $\boldsymbol{\rho(A) = 4}$

[illegible] { 10 to 13 }

[illegible]

>> $R_1 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & 0 & 2 & 6 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$R_2 \rightarrow -2\,R_1 + R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & 4 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$R_3 \rightarrow 1/2 \cdot R_2 + R_3$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \rightarrow -1/2 \cdot C_1 + C_2, \quad C_3 \rightarrow -3/2 \cdot C_1 + C_3, \quad C_4 \rightarrow -1/2 \cdot C_1 + C_4$

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow -2 \cdot C_2 + C_3, \quad C_4 \rightarrow -2 \cdot C_2 + C_4$

$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1/2 \cdot C_1, \ -1/2 \cdot C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ **Thus $\rho(A) = 2$**

---

11. $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

>> $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to 2R_1 + R_2, \ R_3 \to -R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$$

$R_3 \to 2R_2 + R_3, \ R_4 \to -3R_2 + R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$R_4 \to R_3 + R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2 \to -2C_1 + C_2,\ C_3 \to -3C_1 + C_3,\ C_4 \to -C_1 + C_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \to -C_2 + C_3,\ C_4 \to -C_2 + C_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$ and $1/2 \cdot C_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ Thus $\rho(A) = 3$

>> $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$R_3 \to -3R_1 + R_3,\ R_4 \to -R_1 + R_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3,\ R_4 \to -R_2 + R_4$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to -C_1 + C_3,\ C_4 \to -C_1 + C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 3C_2 + C_3,\ C_4 \to C_2 + C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ Thus $\rho(A) = 2$

---

13. $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

>> $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

[We can perform elementary transformations like the earlier problems. However if one can observe that the fourth row is the sum of the first three rows we can as well proceed by subtracting the fourth row from the sum of first three rows]

$$R_4 \to -(R_1+R_2+R_3)+R_4$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to -2R_1+R_2,\ R_3 \to -3R_1+R_3$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[*Preferring to avoid fraction let us perform* $-4R_2+5R_3$ *to make the leading entry in third row zero*]

$$R_3 \to -4R_2+5R_3$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[*It is advisible to multiply third row by* $1/11$] $1/11 \cdot R_3$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2 \to C_1+C_2,\ C_3 \to 2C_1+C_3,\ C_4 \to 4C_1+C_4$$

Now,
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \to -3/5 \cdot C_2+C_3,\ C_4 \to -7/5 \cdot C_2+C_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_4 \to -2/3 \cdot C_3 + C_4$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$1/5 \cdot C_2 \,;\, 1/3 \cdot C_3$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{Thus } \rho(A) = 3$$

---

14. *Find the value [illegible] of k such that [illegible] matrix A may have rank equal to (a) 3 [illegible]*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & k \\ 1 & 4 & 10 & k^2 \end{bmatrix}$$

>> **We have** $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & k \\ 1 & 4 & 10 & k^2 \end{bmatrix}$

$$R_2 \to -R_1 + R_2, \quad R_3 \to -R_1 + R_3$$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & (k-1) \\ 0 & 3 & 9 & (k^2-1) \end{bmatrix}$$

$$R_3 \to -3R_2 + R_3$$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & (k-1) \\ 0 & 0 & 0 & (k^2-3k+2) \end{bmatrix}$$

(a) Rank of A can be 3 if the equivalent form of A has 3 non zero rows.

This is possible if $(k^2 - 3k + 2) \neq 0$

ie., $\quad (k-1)(k-2) \neq 0$ or $k \neq 1$ and $k \neq 2$

Thus $\quad \rho(A) = 3$ **if** $k \neq 1$ **and** $k \neq 2$

(b) Rank of A can be 2 if the equivalent form of A has 2 non zero rows.

This is possible if $k^2 - 3k + 2 = 0$ or $(k-1)(k-2) = 0$

Thus $\rho(A) = 2$ **if** $k = 1$ **or** $k = 2$

---

15. *Find non singular matrices P and Q such that PAQ is in the normal form and hence find the rank of A where*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$$

Note : ***The following is the working procedure.***

- We assume $A = I\,A\,I$ where I is the identity matrix of the befitting order.
- We perform elementary transformations to reduce the matrix A to the normal form.
- The row transformations performed on A will also be performed on the first (prefactor of A) of the identity matrix whereas the column transformations performed on A will be performed on the second of the identity matrix (post factor of A).
- Thus we obtain the normal form of A equal to PAQ where P and Q are nonsingular matrices. We can instantly write $\rho(A)$ also.

Remark : P and Q are not unique

We now solve the given problem.

>> Let $A = I_3\ A\ I_3$ where I_3 represents identity matrix of order 3.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -R_1 + R_2,\ R_3 \to -3R_1 + R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $C_2 \to -C_1 + C_2,\ C_3 \to -C_1 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -C_2 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$-1/2 \cdot C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ,$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}$ **and** $Q = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ **Also** $\rho(A) = 2$

>> The order of A is 3×4 hence we need to take the prefactor of A as I_3 and post factor of A as I_4. That is,

$$A = I_3 \, A \, I_4$$

ie., $$A = \begin{bmatrix} 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \\ 0 & 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & 0 & 2 & 6 \\ 0 & 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -2\,R_1 + R_2$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & 4 \\ 0 & 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 1/2 \cdot R_2 + R_3$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, $C_2 \to -1/2 \cdot C_1 + C_2,\ C_3 \to -3/2 \cdot C_1 + C_3,\ C_4 \to -1/2 \cdot C_1 + C_4$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & -3/2 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -2\,C_2 + C_3,\ C_4 \to 2\,C_2 + C_4$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & -1/2 & -3/2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$1/2 \cdot C_1,\ -1/2 \cdot C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1/2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1/2 & 1/4 & -1/2 & -3/2 \\ 0 & -1/2 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$ **where** $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 1/2 & -1 & 1 \end{bmatrix}$ **and** $Q = \begin{bmatrix} 1/2 & 1/4 & -1/2 & -3/2 \\ 0 & -1/2 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

7.6 Consistency of a system of linear equations

A system of equations in which all the unknowns quantities appear in the first degree alone is called a *linear system of equations.*

A set of m linear equations in n unknown is as follows.

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

where a_{ij}'s and b_i's are constants.

If $b_1, b_2, \ldots b_m$ are all zero, the system is said to be *homogeneous.*

The set of values $x_1, x_2, \ldots x_n$ which satisfy all the equations simultaneously is called a ***solution*** of the system of equations.

A system of linear equations is said to be ***consistent*** if it possess a solution. Otherwise the system is said to be ***inconsistent.***

The above system of equations can be written in the matrix form,

$$\begin{bmatrix} a_{11} & a_{12} & \cdot\cdot & a_{1n} \\ a_{21} & a_{22} & \cdot\cdot & a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_{m1} & a_{m2} & \cdot\cdot & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot\cdot \\ \cdot\cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot\cdot \\ \cdot\cdot \\ b_m \end{bmatrix}$$

That is, $AX = B$ where the matrix A is of order $m \times n$, X is of order $n \times 1$ and hence their product denoted by B is of order $m \times 1$. Obviously $AX = [\,0\,]$ is the matrix representation of the homogeneous set of equations where $[0]$ is a null matrix of order $m \times 1$.

$x_1 = x_2 = x_3 = \ldots = x_n = 0$ is obviously a solution of the homogeneous system of equations and is called a ***trivial solution***. If at least one x_i $(i = 1, 2, \ldots n)$ is not equal to zero then it is called a ***non trivial solution.***

The system of equations $AX = B$ may or may not possess a solution. If the system possesses a solution it may or may not be the only solution. The concept of the rank of a matrix helps us to conclude (i) whether the system is consistent or not (ii) whether the system possess uruque solution or many solutions.

Condition for consistency and types of solution

Consider a system of m equations in n unknowns represented in the matrix form $AX = B$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called the ***coefficient matrix***.

The matrix formed by appending to A an extra column consisting of the elements of B is called the ***augmented matrix*** denoted by $[A : B]$

$$i.e., \quad [A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \cdots & \cdots & \cdots & \cdots & : & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & b_m \end{bmatrix}$$

The system of equations represented by the matrix equation $AX = B$
is consistent if $\rho(A) = \rho[A : B]$

Suppose $\rho[A] = \rho[A : B] = r$ then the condition for various types of solution are as follows.

1. ***Unique solution*** : $\rho[A] = \rho[A : B] = r = n$, n being the number of unknowns.
2. ***Infinite solutions*** : $\rho[A] = \rho[A : B] = r < n$

In this case $(n - r)$ unknowns can take arbitrary values. Obviously $\rho[A] \neq \rho[A : B]$ implies that the system is ***inconsistent***. (*does not possess a solution*)

Working procedure for problems

- We first form the augmented matrix $[A : B]$ and we can clearly identify the portion of the coefficient matrix A in it.
- We reduce the matrix $[A : B]$ to an echelon form by elementary row transformations. This will enable us to immediately write down the ranks of A and also $[A : B]$ with the result we can decide the consistency aspect of the system of equations.
- The echelon form of $[A : B]$ is converted back to the equation form and the solution will emerge easily.

WORKED PROBLEMS

17. *Test for consistency and solve*

[illegible]

[illegible]

[illegible]

>> $$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$$ is the augmented matrix.

We now perform elementary row operations.

$$R_2 \to -R_1 + R_2,\ R_3 \to -3R_1 + R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

$$R_3 \to -R_2 + R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & 3 & : & -9 \end{bmatrix}$$

Note : *We need not make the leading non zero entry in every row 1 as we can decide on the rank of the matrices A and* $[A:B]$ *at this stage.*

Both A and $[A:B]$ matrices have all the three rows non zero.

$\therefore \quad \rho[A] = 3$ and $\rho[A:B] = 3 \quad$ *ie.,* $r = 3$

Also the number of independent variables $n = 3$

Since $\rho[A] = \rho[A:B] = 3$ (*ie.,* $r = n = 3$) **the given system of equations is consistent and will have unique solution.**

Let us now convert the prevailing form of $[A:B]$ into a set of equations as follows.

$$x + y + z = 6 \quad \ldots \text{(i)}$$

$$-2y + z = -1 \quad \ldots \text{(ii)}$$

$$-3z = -9 \quad \ldots \text{(iii)}$$

From (iii) $z = 3$, substituting this value in (ii) we get $y = 2$. Finally substituting these values in (i) we get $x = 1$

Thus $\mathbf{x = 1, y = 2, z = 3}$ is the unique solution.

>> Now $[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 4 & 5 & 7 & : & 35 \\ 3 & 3 & 4 & : & 21 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -4R_1 + R_2, \quad R_3 \to -3R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & -3 & -5 & : & -21 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We have $\rho[A] = 2$, $\rho[A:B] = 2$ *i.e.,* $r = 2$. Also $n = 3$

Since $\rho[A] = \rho[A:B] = 2 < 3$ (*ie.,* $r < n$) **the system is consistent and will have infinite solutions** Here $(n - r) = 1$ and hence one of the variables can take arbitrary values.

We now have, $\quad x + 2y + 3z = 14 \quad \ldots$ (i)

$-3y - 5z = -21 \quad \ldots$ (ii)

Let $z = k$ be arbitrary.

$\therefore$ from (ii) $-3y - 5k = -21$ or $y = \dfrac{21 - 5k}{3} = 7 - \dfrac{5k}{3}$

Now from (i) $x + 2(7 - 5k/3) + 3k = 14 \quad \therefore x = k/3$

Thus $x = \dfrac{k}{3}$, $y = 7 - \dfrac{5k}{3}$, $z = k$ represent infinite solutions since k is arbitrary.

19. *Test for consistency and solve :*

$$x - 4y + 7z = 14$$

$$3x + 8y - 2z = 13$$

$$7x - 8y + 26z = 5$$

>> $[A:B] = \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 3 & 8 & -2 & : & 13 \\ 7 & -8 & 26 & : & 5 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -3R_1 + R_2, \; R_3 \to -7R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 20 & -23 & : & -93 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 0 & 0 & : & -64 \end{bmatrix}$$

We have $\rho[A] = 2$ and $\rho[A:B] = 3$

Since $\rho[A] \neq \rho[A:B]$ the system is **inconsistent** (*does not possess any solution*)

Remark : *By the conversion of the above form of the matrix to equations, the last row will result in an identity of the form* $0 = -64$ *which is absurd. Hence no solution for the system.*

---

20. *Test for consistency and solve :*

$$5x_1 + x_2 + 3x_3 = 20$$

$$2x_1 + 5x_2 + 2x_3 = 18$$

$$3x_1 + 2x_2 + x_3 = 14$$

>> $[A:B] = \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 2 & 5 & 2 & : & 18 \\ 3 & 2 & 1 & : & 14 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -2R_1 + 5R_2, \; R_3 \to -3R_1 + 5R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 7 & -4 & : & 10 \end{bmatrix}$$

$R_3 \to -7R_2 + 23R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 0 & -120 & : & -120 \end{bmatrix}$$

We have $\rho[A] = 3$, $\rho[A:B] = 3$ *ie.,* $r = 3$. Also $n = 3$

Since $\rho[A] = \rho[A:B] = 3$ (*ie.,* $r = n = 3$) **the system is consistent and will have unique solution.**

We now have,
$$5x_1 + x_2 + 3x_3 = 20 \quad \ldots \text{(i)}$$
$$23x_2 + 4x_3 = 50 \quad \ldots \text{(ii)}$$
$$-120x_3 = -120 \quad \ldots \text{(iii)}$$

$\therefore$ from (iii), $x_3 = 1$

From (ii) we get $x_2 = 2$ and from (i) we get $x_1 = 3$

Thus $\mathbf{x_1 = 3,\ x_2 = 2,\ x_3 = 1}$ is the unique solution.

---

21. *Test for consistency and* [illegible]

$$x + 2y + 2z = 1$$
$$2x + y + z = 2$$
$$3x + 2y + 2z = 3$$
$$y + z = 0$$

>> $[A:B] = \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 2 & 1 & 1 & : & 2 \\ 3 & 2 & 2 & : & 3 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -2R_1 + R_2$, $R_3 \to -3R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & -3 & -3 & : & 0 \\ 0 & -4 & -4 & : & 0 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & -4 & -4 & : & 0 \\ 0 & -3 & -3 & : & 0 \end{bmatrix}$$

$R_3 \to 4R_2 + R_3, \; R_4 \to 3R_2 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We have $\rho[A] = 2, \; \rho[A:B] = 2$ *i.e.*, $r = 2$. Also $n = 3$.

Since $\rho[A] = \rho[A:B] = 2 < 3$ (*i.e.*, $r < n$) **the system is consistent and will have infinite solutions.**

We now have, $\quad x + 2y + 2z = 1 \quad \ldots$ (i)

$$y + z = 0 \quad \ldots \text{(ii)}$$

Let $z = k$ be arbitrary $\therefore \; y = -k$ from (ii).

Also from (i), $x + (-2k) + 2k = 1 \;\; \therefore \; x = 1$

Thus $\mathbf{x = 1, \; y = -k, \; z = k}$ are the infinite number of solutions since k is arbitrary.

---

22. *Show that the following system of equations does not possess any solution*

$$5x + 3y + 7z = 5$$

$$3x + 26y + 2z = 9$$

$$7x + 2y + 10z = 5$$

>> $\quad [A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -3R_1 + 5R_2, \; R_3 \to -7R_1 + 5R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & -11 & 1 & : & -10 \end{bmatrix}$$

$R_3 \to R_2 + 11R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & 0 & 0 & : & -80 \end{bmatrix}$$

We have $\rho[A] = 2$, $\rho[A:B] = 3$

Since $\rho[A] \neq \rho[A:B]$ **the system is inconsistent.**

---

23. *Test for consistency and solve*

$5x + 3y + 7z = 4$
$3x + 26y + 2z = 9$
$7x + 2y + 10z = 5$

>> $[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -3R_1 + 5R_2, \; R_3 \to -7R_1 + 5R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 121 & -11 & : & 33 \\ 0 & -11 & 1 & : & -3 \end{bmatrix}$$

$R_2 \to -1/11 \cdot R_2$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & -11 & 1 & : & -3 \end{bmatrix}$$

$R_3 \to R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We have $\rho[A] = 2$, $\rho[A:B] = 2$

Since $\rho[A] = \rho[A:B] = 2 < 3$ we conclude **that the system is consistent and will have infinite solutions.**

We now have, $5x + 3y + 7z = 4$ . . . (i)

$11y - z = 3$ . . . (ii)

Let $z = k$ be arbitrary and from (ii) $y = \frac{1}{11}(k+3)$

Also from (i), $5x + \frac{3}{11}(k+3) + 7k = 4 \quad \therefore \; x = \frac{1}{11}(7 - 16k)$

Thus $\quad x = \frac{1}{11}(7 - 16k), \; y = \frac{1}{11}(k+3), \; z = k$ is the required solution.

24. [illegible]

[illegible]

>> $$[A:B] = \begin{bmatrix} 4 & -5 & 1 & : & -3 \\ 2 & 3 & -1 & : & 3 \\ 3 & -1 & 2 & : & 5 \\ 1 & 2 & -5 & : & -9 \end{bmatrix}$$ is the augmented matrix

$R_1 \leftrightarrow R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & -5 & : & -9 \\ 2 & 3 & -1 & : & 3 \\ 3 & -1 & 2 & : & 5 \\ 4 & -5 & 1 & : & -3 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2, \; R_3 \to -3R_1 + R_3, \; R_4 \to -4R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & -5 & : & -9 \\ 0 & -1 & 9 & : & 21 \\ 0 & -7 & 17 & : & 32 \\ 0 & -13 & 21 & : & 33 \end{bmatrix}$$

$R_3 \to -7R_2 + R_3, \; R_4 \to -13R_2 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & -5 & : & -9 \\ 0 & -1 & 9 & : & 21 \\ 0 & 0 & -46 & : & -115 \\ 0 & 0 & -96 & : & -240 \end{bmatrix}$$

$-1/23 \cdot R_3, \; -1/48 \cdot R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & -5 & : & -9 \\ 0 & -1 & 9 & : & 21 \\ 0 & 0 & 2 & : & 5 \\ 0 & 0 & 2 & : & 5 \end{bmatrix}$$

$R_4 \to -R_3 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & -5 & : & -9 \\ 0 & -1 & 9 & : & 21 \\ 0 & 0 & 2 & : & 5 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We have $\rho[A] = 3$, $\rho[A:B] = 3$ *i.e.,* $r = 3$, also $n = 3$.

Since $\rho[A] = \rho[A:B] = 3$ (*i.e.,* $r = n = 3$) **the system is consistent and will have unique solution.**

We now have

$$x + 2y - 5z = -9 \quad \ldots \text{(i)}$$

$$-y + 9z = 21 \quad \ldots \text{(ii)}$$

$$2z = 5 \quad \ldots \text{(iii)}$$

From (iii), $z = \frac{5}{2}$ $\therefore$ (ii) becomes $y = \frac{45}{2} - 21$ or $y = \frac{3}{2}$

Also from (i), $x + 3 - \frac{25}{2} = -9$ $\therefore$ $x = \frac{1}{2}$

Thus $x = \frac{1}{2}$, $y = \frac{3}{2}$, $z = \frac{5}{2}$ is the unique solution.

---

25. *Investigate the values of λ and μ such that the system of equations*

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu, \text{ may have}$$

(a) *Unique solution* (b) *Infinite solution* (c) *No solution*

>> $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$ is the augmented matrix

$R_2 \to -R_1 + R_2$, $R_3 \to -R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$$

(a) *Unique solution* : We must have $\rho[A] = \rho[A:B] = 3$, $\rho[A]$ will be 3 if $(\lambda-3) \neq 0$ since the other two entries in the last row of A are zero. If $(\lambda-3) \neq 0$ or $\lambda \neq 3$ irrespective of the value of μ, $\rho[A:B]$ will also be 3.

$\therefore$ the system will have **unique solution if $\lambda \neq 3$**

(b) *Infinite solutions* : Here we have $n = 3$ and we need $\rho[A] = \rho[A:B] = r < 3$. We must have $r = 2$ since first row and second row are non zero.

$\therefore$ $\rho[A] = \rho[A:B] = 2$ only when the last row of $[A:B]$ is completely zero. This is possible if $\lambda - 3 = 0$, $\mu - 10 = 0$

$\therefore$ the system will have **infinite solution if $\lambda = 3$ and $\mu = 10$**

(c) *No solution* : We must have $\rho[A] \neq \rho[A:B]$. By case (a) $\rho[A] = 3$ if $\lambda \neq 3$ and hence if $\lambda = 3$ we obtain $\rho[A] = 2$.
If we impose $(\mu - 10) \neq 0$ then $\rho[A:B]$ will be 3.

$\therefore$ the system has **no solution if $\lambda = 3$ and $\mu \neq 10$**

---

26. *Find for what values of k the system of equations*

$$x + y + z = 1$$

$$x + 2y + 4z = k$$

$$x + 4y + 10z = k^2,$$

possesses a solution. Solve completely in each case.

>> $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & k \\ 1 & 4 & 10 & : & k^2 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -R_1 + R_2$, $R_3 \to -R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & k-1 \\ 0 & 3 & 9 & : & k^2-1 \end{bmatrix}$$

$$R_3 \rightarrow -3R_2 + R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & k-1 \\ 0 & 0 & 0 & : & k^2-3k+2 \end{bmatrix}$$

$\rho[A] = 2$ and for the system to be consistent we must have $\rho[A:B]$ also 2. This is possible if $k^2 - 3k + 2 = 0 \quad \therefore \quad k = 1,\ k = 2.$

Hence we conclude that the system possesses a solution if $k = 1, 2$.

Since $\rho[A] = \rho[A:B] = 2 < 3$ for the cases $k = 1, 2$ the system will have infinite solution and the same are as follows.

Case - i : $k = 1$. The system of equations are,

$$x + y + z = 1 \quad \ldots \text{(i)}$$

$$y + 3z = 0 \quad \ldots \text{(ii)}$$

Let $z = k_1$ be arbitrary $\therefore$ from (ii) $y = -3k_1$ and from (i) $x = 1 + 2k_1$

Case - ii: $k = 2$. The system of equations are

$$x + y + z = 1 \quad \ldots \text{(iii)}$$

$$y + 3z = 1 \quad \ldots \text{(iv)}$$

Let $z = k_2$ be arbitrary $\therefore$ from (iv) $y = 1 - 3k_2$ and from (iii) $x = 2k_2$

Thus $x = 1 + 2k_1$, $y = -3k_1$, $z = k_1$ and $x = 2k_2$, $y = 1 - 3k_2$, $z = k_2$ give all the solution of the given system of equations.

27. Show that the system of equations

$$3x + 4y + 5z = a$$

$$4x + 5y + 6z = b$$

$$5x + 6y + 7z = c$$

possesses a solution if $a + c = 2b$ [illegible] $(a, b, [illegible]) = (1, 2, 3)$

>> $[A:B] = \begin{bmatrix} 3 & 4 & 5 & : & a \\ 4 & 5 & 6 & : & b \\ 5 & 6 & 7 & : & c \end{bmatrix}$ is the augmented matrix.

Observing the elements in A, we perform the transformations : $R_3 \to -R_2 + R_3$; $R_2 \to -R_1 + R_2$ for convenience (*We can as well perform* $R_2 \to -4R_1 + 3R_3$ *and* $R_3 \to -5R_1 + 3R_3$)

$$[A:B] \sim \begin{bmatrix} 3 & 4 & 5 & : & a \\ 1 & 1 & 1 & : & b-a \\ 1 & 1 & 1 & : & c-b \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & b-a \\ 3 & 4 & 5 & : & a \\ 1 & 1 & 1 & : & c-b \end{bmatrix}$$

$R_2 \to -3R_1 + R_2$, $R_3 \to -R_1 + R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & b-a \\ 0 & 1 & 2 & : & 4a-3b \\ 0 & 0 & 0 & : & a-2b+c \end{bmatrix}$$

Here $\rho[A] = 2$ and the system possesses a solution if $\rho[A:B]$ is also equal to 2 which is possible only when $a - 2b + c = 0$ or $a + c = 2b$ as required.

We shall solve the system when $a = 1$, $b = 2$, $c = 3$

For these values we have, $3x + 4y + 5z = 1$ ... (i)

$y + 2z = -2$ ... (ii)

Let $z = k$ be arbitrary. From (ii) $y = -(2 + 2k)$

From (i): $3x - 8 - 8k + 5k = 1$ $\therefore$ $x = k + 3$

Thus $x = k+3$, $y = -2(1+k)$, $z = k$ is the required solution when $(a, b, c) = (1, 2, 3)$

Solution of linear homogeneous equations

Let $AX = 0$ respresent the matrix representation of m linear homogeneous equations in n variables. The augmented matrix is given by

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & : & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & : & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & : & 0 \end{bmatrix}$$

If $\rho[A] = \rho[A:B] = n$, the system is consistent and will have **trivial solution** $x_1 = 0 = x_2 = x_3 = \cdots = x_n$

If $\rho[A] = \rho[A:B] = r < n$ the system will have **non trivial, infinite number of solutions** as $(n - r)$ variables can be chosen arbitrarily.

WORKED PROBLEMS

[illegible]

$$x + y + 3z = 0$$

$$4x + 3y + \lambda z = 0$$

$$2x + y + 2z = 0$$

>> $[A:B] = \begin{bmatrix} 1 & 1 & 3 & : & 0 \\ 4 & 3 & \lambda & : & 0 \\ 2 & 1 & 2 & : & 0 \end{bmatrix}$ is the augmented matrix.

$R_2 \to -4R_1 + R_2,\ R_3 \to -2R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 3 & : & 0 \\ 0 & -1 & (\lambda - 12) & : & 0 \\ 0 & -1 & -4 & : & 0 \end{bmatrix},$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 3 & : & 0 \\ 0 & -1 & (\lambda - 12) & : & 0 \\ 0 & 0 & (8 - \lambda) & : & 0 \end{bmatrix}$$

Here $n = 3$ and we need to have $\rho[A] = \rho[A:B] = r < 3$

If $8 - \lambda = 0$, then $\rho[A] = \rho[A:B] = 2 < 3$

Thus when $\lambda = 8$, the system will have non trivial solution.

Now, when $\lambda = 8$ we have the system of equations

$$x + y + 3z = 0 \quad \ldots \text{(i)}$$

$$-y - 4z = 0 \quad \ldots \text{(ii)}$$

Let $z = k$ be arbitrary $\therefore$ $y = -4k$ from (ii).

Also from (i), $x - 4k + 3k = 0$ $\therefore$ $x = k$

Thus $x = k$, $y = -4k$ **and** $z = k$ **represents the non trivial solution** of the given system of equations when $\lambda = 8$.

29. *Find the value of k such that the following system of equations possess a nontrivial solution. Also find the solution of the system.*

$$4x_1 + 9x_2 + x_3 = 0$$

$$kx_1 + 3x_2 + kx_3 = 0$$

$$x_1 + 4x_2 + 2x_3 = 0$$

>> $[A:B] = \begin{bmatrix} 4 & 9 & 1 & : & 0 \\ k & 3 & k & : & 0 \\ 1 & 4 & 2 & : & 0 \end{bmatrix}$ is the augmented matrix.

$R_1 \leftrightarrow R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 4 & 2 & : & 0 \\ k & 3 & k & : & 0 \\ 4 & 9 & 1 & : & 0 \end{bmatrix}$$

$R_2 \to -kR_1 + R_2, \; R_3 \to -4R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 4 & 2 & : & 0 \\ 0 & (3-4k) & -k & : & 0 \\ 0 & -7 & -7 & : & 0 \end{bmatrix}$$

$-1/7 \cdot R_3$ and then $R_2 \leftrightarrow R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 4 & 2 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & (3-4k) & -k & : & 0 \end{bmatrix}$$

$R_3 \to (4k-3)\, R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 4 & 2 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & (3k-3) & : & 0 \end{bmatrix}$$

$\rho[A] = \rho[A:B] = 2 < 3$, only when $3k-3 = 0$ or $k = 1$

Thus when $k = 1$, the system will have non trivial solution.

Now, when $k = 1$, we have the system of equations,

$$x + 4y + 2z = 0 \qquad \ldots \text{(i)}$$

$$y + z = 0 \qquad \ldots \text{(ii)}$$

Let $z = k$ be arbitrary. $\therefore$ $y = -k$ from (ii).

Also from (i), $x - 4k + 2k = 0$ $\therefore$ $x = 2k$

Thus $x = 2k$, $y = -k$ and $z = k$, is the required non trivial solution.

$$>> \quad [A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 2 & 3 & 1 & : & 0 \\ 4 & 5 & 4 & : & 0 \\ 1 & 1 & -2 & : & 0 \end{bmatrix} \text{ is the augmented matrix.}$$

$R_2 \to -2R_1 + R_2,\ R_3 \to -4R_1 + R_3,\ R_4 \to -R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 0 & -1 & -5 & : & 0 \\ 0 & -3 & -8 & : & 0 \\ 0 & -1 & -5 & : & 0 \end{bmatrix}$$

$R_3 \to -3R_2 + R_3,\ R_4 \to -R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 0 & -1 & -5 & : & 0 \\ 0 & 0 & 7 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho[A] = \rho[A:B] = 3 =$ number of un knowns.

Hence the system is consistent and will have **trivial solution** :
$x = 0,\ y = 0,\ z = 0$

$$>> \quad [A:B] = \begin{bmatrix} 1 & 1 & -1 & 1 & : & 0 \\ 1 & -1 & 2 & -1 & : & 0 \\ 3 & 1 & 0 & 1 & : & 0 \end{bmatrix}$$

$R_2 \to -R_1 + R_2,\ R_3 \to -3R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & -1 & 1 & : & 0 \\ 0 & -2 & 3 & -2 & : & 0 \\ 0 & -2 & 3 & -2 & : & 0 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & -1 & 1 & : & 0 \\ 1 & -2 & 3 & -2 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho[A] = \rho[A:B] = 2 < 4$, 4 being the number of unknowns.

The system does possess non trivial solution. $(4-2) = 2$ unknowns can be chosen arbitrarily.

We now have the system of equations,

$$x_1 + x_2 - x_3 + x_4 = 0 \qquad \ldots \text{(i)}$$

$$-2x_2 + 3x_3 - 2x_4 = 0 \qquad \ldots \text{(ii)}$$

Let $x_4 = k_1$ and $x_3 = k_2$ be arbitrary.

From (ii), $-2x_2 + 3k_2 - 2k_1 = 0 \therefore x_2 = \frac{1}{2}(3k_2 - 2k_1)$

From (i), $x_1 + \frac{1}{2}(3k_2 - 2k_1) - k_2 + k_1 = 0 \therefore x_1 = -\frac{1}{2}k_2$

Thus $x_1 = -\frac{1}{2}k_2,\ x_2 = \frac{3}{2}k_2 - k_1,\ x_3 = k_2,\ x_4 = k_1$ is the required non trivial solution.

>> $$[A:B] = \begin{bmatrix} 1 & -2 & 1 & -1 & : & 0 \\ 1 & 1 & -2 & 3 & : & 0 \\ 4 & 1 & -5 & 8 & : & 0 \\ 5 & -7 & 2 & -1 & : & 0 \end{bmatrix}$$ is the augmented matrix.

$R_2 \to -R_1 + R_2,\ R_3 \to -4R_1 + R_3,\ R_4 \to -5R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & -2 & 1 & -1 & : & 0 \\ 0 & 3 & -3 & 4 & : & 0 \\ 0 & 9 & -9 & 12 & : & 0 \\ 0 & 3 & -3 & 4 & : & 0 \end{bmatrix}$$

$R_3 \to -3R_2 + R_3,\ R_4 \to -R_2 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & -2 & 1 & -1 & : & 0 \\ 0 & 3 & -3 & 4 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We have $\rho[A:B] = 2$ and $\rho[A] = 2$ *i.e.,* $r = 2$. Also $n = 4$.

Since $\rho[A:B] = \rho[A] = 2 < 4$ the system is consistent and will have infinite solutions by choosing $(n-r)$ variables (*i.e.,* 2 variables) arbitrarily.

We now have

$$x_1 - 2x_2 + x_3 - x_4 = 0 \qquad \ldots \text{(i)}$$

$$3x_2 - 3x_3 + 4x_4 = 0 \qquad \ldots \text{(ii)}$$

Let us choose $x_4 = k_1,\ x_3 = k_2$ arbitrarily.

$\therefore$ from (ii), $3x_2 - 3k_2 + 4k_1 = 0$ or $x_2 = k_2 - \frac{4}{3}k_1$

Also from (i), $x_1 - 2k_2 + \frac{8}{3}k_1 + k_2 - k_1 = 0$ or $x_1 = k_2 - \frac{5}{3}k_1$

Thus $x_1 = k_2 - \frac{5}{3}k_1,\ x_2 = k_2 - \frac{4}{3}k_1,\ x_3 = k_2,\ x_4 = k_1$ give all the solutions of the given system of equations.

Find the rank of the following matrices by elementary row transformations.

1. $\begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 3 & 2 & 5 & 2 \\ 2 & -1 & 1 & -3 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \\ 2 & 1 & 5 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & 5 & -4 & 7 \\ -1 & -2 & -1 & 2 \\ 3 & 3 & -5 & 10 \end{bmatrix}$ **6.** $\begin{bmatrix} 8 & 2 & 1 & 6 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 3 \\ 5 & 1 & 1 & 4 \end{bmatrix}$

Find the rank of the following matrices by reducing it to the normal form

7. $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 1 & 3 & 3 & 11 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 2 & 1 & 4 \\ 1 & 4 & 3 & 2 \\ 4 & 6 & 4 & 6 \\ 7 & 8 & 5 & 10 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 4 & 6 \\ 3 & 5 & 6 & 10 \\ 4 & -1 & -3 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

Test for consistency and solve the following system of equations

11. $x+y+z = 9$, $2x+5y+7z = 52$, $2x+y-z = 0$

12. $x-y+2z = 3$, $x+2y+3z = 5$, $3x+4y-5z = -13$

13. $5x+y+3z = 20$, $2x+5y+2z = 18$, $3z+2y+z = 14$

14. $x+3y-2z = 0$, $2x-y+4z = 0$, $x-11y+14z = 0$

15. $4x-2y+6z = 8$, $x+y-3z = -1$, $15x-3y+9z = 21$

16. $x+y+z = 1$, $x+2y+3z = 4$, $x+3y+5z = 7$, $x+4y+7z = 10$

17. $2x+6y+11 = 0$, $6x+20y-6z+3 = 0$, $6y-18z+1 = 0$

18. $2x-3y+7z = 5$, $3x+y-3z = 13$, $2x+19y-47z = 32$

19. Find the values of λ and μ such that the following system of equations, $2x+3y+5z = 9$, $7x+3y-2z = 8$, $2x+3y+\lambda z = \mu$ may have (a) Unique solution (b) Many solution (c) No solution.

20. Show that the equations, $-2x+y+z = a$, $x-2y+z = b$, $x+y-2z = c$ is consistent if $a+b+c = 0$. Solve the system of equations when $(a,b,c) = (1,1,-2)$.

ANSWERS

1. 3	**2.** 3	**3.** 2	**4.** 2	**5.** 3
6. 3	**7.** 3	**8.** 2	**9.** 3	**10.** 2

11. $x = 1,\ y = 3,\ z = 5$ **12.** $x = -1, y = 0, z = 2$

13. $x = 3,\ y = 2,\ z = 1$ **14.** $x = -10k/7,\ y = 8k/7,\ z = k$

15. $x = 1,\ y = 3k-2,\ z = k$ **16.** $x = k-2,\ y = 3-2k,\ z = k$

17. Inconsistent **18.** Inconsistent

19. (a) Unique solution : $\lambda \neq 5,$
(b) Infinite solution : $\lambda = 5,\ \mu = 9$
(c) No solution : $\lambda = 5,\ \mu \neq 9$

20. $x = y = k-1,\ z = k$

Solution of a system of non homogeneous equations

Gauss elimination method

This method is illustrated by considering a system of three independent equations in three unknowns. The method is very much similar to the method we employed in solving a system of equations by testing its consistency.

Consider the system of equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad \ldots \text{(i)}$$

The system (i) is equivalent to the matrix equation

$$AX = B \quad \ldots \text{(ii)}$$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The method aims in reducing the coefficient matrix A to an upper triangular matrix.

We consider the augmented matrix $[A : B]$

$$\text{We have } [A : B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{bmatrix}$$

Step - 1. We use the element a_{11} ($\neq 0$) to make the elements a_{21} and a_{31} zero by elementary row transformations.

This transforms [$A : B$] into the form

$$[A : B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & a_{22}' & a_{23}' & : & b_2' \\ 0 & a_{32}' & a_{33}' & : & b_3' \end{bmatrix} \quad , \qquad \ldots \text{(iii)}$$

Step - 2 We use the element a_{22}' ($\neq 0$) to make the element a_{32}' zero by elementary row transformation.

This transforms [$A : B$] as in (iii) into the form

$$[A : B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & a_{22}' & a_{23}' & : & b_2' \\ 0 & 0 & a_{33}'' & : & b_3'' \end{bmatrix} \qquad \ldots \text{(iv)}$$

(*Observe that the matrix A is reduced to the upper triangular form*)

From (iv), the given system of linear equations is equivalent to the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}'x_2 + a_{23}'x_3 &= b_2' \\ a_{33}''x_3 &= b_3'' \end{aligned}$$

We get x_3 from the last of the equation and by back substitution we get x_2 and x_1. The values of x_1, x_2, x_3 so obtained constitutes the exact solution of the given system of equations.

Note : *In case* a_{11} *or* a_{22}' *is zero we only need to re-arrange the equations.*

This method can be regarded as the modification of Gauss elimination method.

This method aims in reducing the coefficient matrix A to a diagonal matrix.

Step-1 This step is same as in Gauss elimination method.

Step-2 Referring to (iii), we use the element a_{22}' ($\neq 0$) to make the elements a_{12} and a_{32}' zero by elementary row transformations.

With this transformation we obtain

$$[A:B] \sim \begin{bmatrix} a_{11} & 0 & a_{13}' & : & b_1' \\ 0 & a_{22}' & a_{23}' & : & b_2' \\ 0 & 0 & a_{33}'' & : & b_3'' \end{bmatrix} \qquad \ldots (v)$$

Step-3 Finally we use the element $a_{33}''\ (\neq 0)$ to make the elements a_{23}', a_{13}' zero by elementary row transformations.

With this transformation we obtain

$$[A:B] \sim \begin{bmatrix} a_{11} & 0 & 0 & : & b_1'' \\ 0 & a_{22}' & 0 & : & b_2'' \\ 0 & 0 & a_{33}'' & : & b_3'' \end{bmatrix} \qquad \ldots (vi)$$

(*Observe that the matrix* A *is reduced to the diagonal form*)

It can be easily seen from (vi) that

$$a_{11}x_1 = b_1'',\ a_{22}'x_2 = b_2'',\ a_{33}''x_3 = b_3''$$

Thus we get the required x_1, x_2, x_3 being the exact solution.

WORKED PROBLEMS

33. [illegible]

$$x + y + z = 9$$
$$x - 2y + 3z = 8$$
$$2x + y - z = 3$$

>> **(a)** *By Gauss elimination method* :

The augmented matrix of the system is

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 1 & -2 & 3 & : & 8 \\ 2 & 1 & -1 & : & 3 \end{bmatrix}$$

$R_2 \to -R_1 + R_2$, $R_3 \to -2R_1 + R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & -1 & -3 & : & -15 \end{bmatrix},$$

$R_3 \to R_2 + (-3)R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & 11 & : & 44 \end{bmatrix}$$

Hence we have , $x + y + z = 9$

$-3y + 2z = -1$

$11z = 44 \quad \therefore \ z = 4$

By back substitution , $-3y + 8 = -1 \quad \therefore \ y = 3$; Also $x = 2$

Thus $x = 2, \ y = 3, \ z = 4$ is the required solution.

(b) *By Gauss Jordan method :*
The first step is same as in Gauss elimination method. So we have

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & -1 & -3 & : & -15 \end{bmatrix}$$

We use the leading non zero entry in second row (– 3) to make the element above (1) and below (– 1) zero.

$$R_1 \to R_2 + 3R_1, \ R_3 \to R_2 + (-3) R_3$$

$$[A:B] \sim \begin{bmatrix} 3 & 0 & 5 & : & 26 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & 11 & : & 44 \end{bmatrix}$$

$R_3 \to 1/11 \cdot R_3$ (*This step is performed only for convenience to get 1 as the leading entry in the third row*).

$$[A:B] \sim \begin{bmatrix} 3 & 0 & 5 & : & 26 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

[*We use the element 1 in the third row 1 to make the two elements above (2 and 5) zero.*]

$$R_1 \to -5R_3 + R_1, \ R_2 \to -2R_3 + R_2$$

$$[A:B] \sim \begin{bmatrix} 3 & 0 & 0 & : & 6 \\ 0 & -3 & 0 & : & -9 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

Hence we have $3x = 6, \ -3y = -9, \ z = 4$

Thus $x = 2, \ y = 3, \ z = 4$ is the required solution.

>> The augmented matrix of the system is

$$[A:B] = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 4 & 11 & -1 & : & 33 \\ 8 & -3 & 2 & : & 20 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2,\ R_3 \to -4R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & -7 & -14 & : & -28 \end{bmatrix}$$

$1/9 \cdot R_2,\ -1/7 \cdot R_3$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 1 & -1 & : & 1 \\ 0 & 1 & 2 & : & 4 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 1 & -1 & : & 1 \\ 0 & 0 & 3 & : & 3 \end{bmatrix}$$

Hence we have, $2x + y + 4z = 12$

$y - z = 1$

$3z = 3 \quad \therefore\ z = 1$

By back substitution, $y = 2$ and $x = 3$

Thus $\mathbf{x = 3,\ y = 2,\ z = 1}$ is the required solution.

>> *It is convenient to perform row transformations if the leading entry is 1. We shall write the augmented matrix by interchanging the first equation with the fourth equation. Fourth equation has three of its coefficients 1.*

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 1 & 7 & 1 & 1 & : & 12 \\ 1 & 1 & 6 & 1 & : & -5 \\ 5 & 1 & 1 & 1 & : & 4 \end{bmatrix}$$

$R_2 \to -R_1 + R_2,\ R_3 \to -R_1 + R_3,\ R_4 \to -5R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 & : & 6 \\ 0 & 6 & 0 & -3 & : & 18 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix}$$

$1/3 \cdot R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 & : & 6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix}$$

$R_4 \to 2R_2 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 & : & 6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & -4 & -21 & : & 46 \end{bmatrix}$$

$R_4 \to 4R_3 + 5R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 & : & 6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & 0 & -117 & : & 234 \end{bmatrix}$$

Hence we have, $x_1 + x_2 + x_3 + 4x_4 = -6$

$$2x_2 + 0x_3 - x_4 = 6$$

$$5x_3 - 3x_4 = 1$$

$$-117x_4 = 234 \quad \therefore\ x_4 = -2$$

By back substitution we get $x_3 = -1,\ x_2 = 2,\ x_1 = 1$

Thus $\boldsymbol{x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2}$ is the required solution.

>> As it is convenient to have the leading coefficient as 1 we shall interchange the first and third equations. The augmented matrix will be

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2,\ R_3 \to -2R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$R_1 \to R_2 + R_1,\ R_3 \to 3R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

$-1/4 \cdot R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_1 \to 2R_3 + R_1,\ R_2 \to 3R_3 + R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & -1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Hence we have $x = 1,\ -y = -3,\ z = 5$

Thus $x = 1,\ y = 3,\ z = 5$ is the required solution.

>> The augmented matrix associated with the given system is

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 2 & -1 & 2 & -1 & : & -5 \\ 3 & 2 & 3 & 4 & : & 7 \\ 1 & -2 & -3 & 2 & : & 5 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2,\ R_3 \to -3R_1 + R_3,\ R_4 \to -R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & -3 & 0 & -3 & : & -9 \\ 0 & -1 & 0 & 1 & : & 1 \\ 0 & -3 & -4 & 1 & : & 3 \end{bmatrix}$$

$-1/3 \cdot R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & 0 & 1 & : & 3 \\ 0 & -1 & 0 & 1 & : & 1 \\ 0 & -3 & -4 & 1 & : & 3 \end{bmatrix}$$

$R_1 \to -R_2 + R_1,\ R_3 \to R_2 + R_3,\ R_4 \to 3R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 1 & 0 & : & -1 \\ 0 & 1 & 0 & 1 & : & 3 \\ 0 & 0 & 0 & 2 & : & 4 \\ 0 & 0 & -4 & 4 & : & 12 \end{bmatrix}$$

We shall perform $1/2 \cdot R_3$, $1/4 \cdot R_4$ and then interchange the rows R_3 and R_4

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 1 & 0 & : & -1 \\ 0 & 1 & 0 & 1 & : & 3 \\ 0 & 0 & -1 & 1 & : & 3 \\ 0 & 0 & 0 & 1 & : & 2 \end{bmatrix}$$

$R_2 \to -R_4 + R_2,\ R_3 \to -R_4 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 1 & 0 & : & -1 \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & -1 & 0 & : & 1 \\ 0 & 0 & 0 & 1 & : & 2 \end{bmatrix}$$

$R_1 \to R_3 + R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & -1 & 0 & : & 1 \\ 0 & 0 & 0 & 1 & : & 2 \end{bmatrix}$$

Hence we have $x_1 = 0,\ x_2 = 1,\ -x_3 = 1,\ x_4 = 2$

Thus $\mathbf{x_1 = 0,\ x_2 = 1,\ x_3 = -1,\ x_4 = 2}$ is the required solution.

>> The augmented matrix associated with the given system of equations is

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 3 & -1 & 4 & : & 4 \\ 2 & 1 & -2 & : & 5 \end{bmatrix}$$

$R_2 \to -3R_1 + R_2,\ R_3 \to -2R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 0 & 5 & -5 & : & -2 \\ 0 & 5 & -8 & : & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 0 & 5 & -5 & : & -2 \\ 0 & 0 & -3 & : & 3 \end{bmatrix}$$

Hence we have, $x_1 - 2x_2 + 3x_3 = 2$

$5x_2 - 5x_3 = -2$

$-3x_3 = 3 \quad \therefore \quad x_3 = -1$

By back substitution $5x_2 + 5 = -2 \quad \therefore \quad x_2 = -7/5 = -1.4$

Also, $x_1 - 2(-1.4) - 3 = 2 \quad \therefore \quad x_1 = 2.2$

Thus $\mathbf{x_1 = 2.2,\ x_2 = -1.4,\ x_3 = -1}$ is the required solution.

>> As it is convenient to have the leading coefficient as 1 we shall interchange the first and third equations. The augmented matrix will be

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2$, $R_3 \to -2R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$R_3 \to 3R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

Hence we have, $x + y + z = 9$

$-y - 3z = -18$

$-4z = -20 \quad \therefore \; z = 5$

By back substitution $y = 3$ and $x = 1$

Thus $x = 1, \; y = 3, \; z = 5$ is the required solution.

>> The augmented matrix associated with the given system is

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 2 & -1 & 2 & -1 & : & -5 \\ 3 & 2 & 3 & 4 & : & 7 \\ 1 & -2 & -3 & 2 & : & 5 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2$, $R_3 \to -3R_1 + R_3$, $R_4 \to -R_1 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & -3 & 0 & -3 & : & -9 \\ 0 & -1 & 0 & 1 & : & 1 \\ 0 & -3 & -4 & 1 & : & 3 \end{bmatrix}$$

$R_3 \to 1/3 \cdot R_2 + R_3,\ R_4 \to -R_2 + R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & -3 & 0 & -3 & : & -9 \\ 0 & 0 & 0 & 2 & : & 4 \\ 0 & 0 & -4 & 4 & : & 12 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & -3 & 0 & -3 & : & -9 \\ 0 & 0 & -4 & 4 & : & 12 \\ 0 & 0 & 0 & 2 & : & 4 \end{bmatrix}$$

Hence we have,

$$x_1 + x_2 + x_3 + x_4 = 2$$
$$-3x_2 + 0x_3 - 3x_4 = -9$$
$$-4x_3 + 4x_4 = 12$$
$$2x_4 = 4 \qquad \therefore\ x_4 = 2$$

Further by back substitution we get $x_3 = -1,\ x_2 = 1$ and $x_1 = 0$

Thus $\mathbf{x_1 = 0,\ x_2 = 1,\ x_3 = -1,\ x_4 = 2}$ is the required solution.

Note : *In many practical situations we come across with more number of equations involving decimal quantities also. In such cases we adopt sequential steps which can be programmed, as illustrated in the following problem. We donot express the equations in the matrix form also.*

An Illustrative Example on Gauss elimination method.

$$0.2\,x + 0.3\,y - 0.1\,z = 0.5$$
$$0.4\,x + 0.4\,y - 0.3\,z = 0.3$$
$$0.2\,x - 0.3\,y + 0.2\,z = 0.2$$

>> **Step-1** The coefficient of x in all the three equations are made 1 by dividing the equations respectively by 0.2 , 0.4 and 0.2. We get,

$$x + 1.5\,y - 0.5\,z = 2.5$$
$$x + y - 0.75\,z = 0.75$$
$$x - 1.5\,y + z = 1$$

Step-2 We subtract the second and third equations from the first equation. We get,

$$x + 1.5\,y - 0.5\,z = 2.5$$
$$0.5\,y + 0.25\,z = 1.75$$
$$3\,y - 1.5\,z = 1.5$$

Step-3 The coefficient of y in the second and third equations are made 1 by dividing the equations respectively by 0.5 and 3. We get,

$$x + 1.5\,y - 0.5\,z = 2.5$$

$$y + 0.5\,z = 3.5$$

$$y - 0.5\,z = 0.5$$

Step-4 We subtract the third equation from the second equation. We get

$$x + 1.5\,y - 0.5\,z = 2.5$$
$$y + 0.5\,z = 3.5$$
$$z = 3$$

By back substitution we get $y = 2$ and $x = 1$

Thus $x = 1,\ y = 2,\ z = 3$ is the required solution.

Remark : *In the event of getting more decimals in the process of division, we can fix the constants to suitable/desired number of decimal places.*

Solve the following system of equations by Gauss elimination method and also by Gauss - Jordan method.

1. $x + 2y + z = 3,\ 2x + 3y + 3z = 10,\ 3x - y + 2z = 13$

2. $2x + 3y - z = 5,\ 4x + 4y - 3z = 3,\ 2x - 3y + 2z = 2$

3. $3x + 4y + 5z = 18,\ 2x - y + 8z = 13,\ 5x - 2y + 7z = 20$

4. $2x_1 - x_2 + x_3 = -1,\ 2x_2 - x_3 + x_4 = 1$

   $x_1 + 2x_3 - x_4 = -1,\ x_1 + x_2 + 2x_4 = 3$

5. $5x_1 + x_2 + x_3 + x_4 = 4,\ x_1 + 7x_2 + x_3 + x_4 = 12$

   $x_1 + x_2 + 6x_3 + x_4 = -5,\ x_1 + x_2 + x_3 + 4x_4 = -6$

Note : *Answers are given in the respective order of the variables as in the given equations*

1. 2, −1, 3   2. 1, 2, 3   3. 3, 1, 1

4. −1, 0, 1, 2   5. 1, 2, −1, −2